Operations with Radicals

Multiplication Property of Radicals: $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{a \cdot b}$, where a and b are real numbers such that $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are also real numbers.

1. Multiply the radicals. Always simplify when you can.

$\sqrt{5} \cdot \sqrt{5}$

$= \sqrt{5 \times 5}$

$= \sqrt{25}$

$= 5$

$\sqrt[3]{7} \cdot \sqrt[3]{3}$

$= \sqrt[3]{21}$

$\sqrt[3]{2} \cdot \sqrt[3]{12}$

$= \sqrt[3]{24}$

$\sqrt{5x^3} \cdot \sqrt{10x^4}$

$= \sqrt{50x^7}$

$= \sqrt{2 \times 25 \cdot x^6 \cdot x}$

$= 5x^3 \cdot \sqrt{2x}$

$(\sqrt{3ab^2})(\sqrt{21a^2b})$

$= \sqrt{63a^3b^3}$

$= \sqrt{9 \times 7 \cdot a^2 \cdot a \cdot b^2 \cdot b}$

$= 3ab\sqrt{7ab}$

$\sqrt[3]{m^2n^2} \cdot \sqrt[3]{48m^4n^2}$

$= \sqrt[3]{6 \times 8 \cdot m^6 \cdot n^4}$

$= \sqrt[3]{6 \times 2^3 \cdot m^6 n^3 \cdot n}$

$= 2m^2n\sqrt{6n}$

$(4\sqrt{3xy^3})(-2\sqrt{6x^3y^2})$

$= -8\sqrt{18x^4y^5}$

$= -8\sqrt{2 \times 9\, x^4 y^4 y}$

$= -24x^2y^2\sqrt{2y}$

2. Notice that $\sqrt{5} \cdot \sqrt{5} = (\sqrt{5})^2 = 5$, and generally, $\sqrt{a} \cdot \sqrt{a} = (\sqrt{a})^2 = a$ when $a \geq 0$. We can even extend this to any real root, by saying, If $\sqrt[n]{a}$ is a real number, then $(\sqrt[n]{a})^n = a$. Using this idea, try the following, and again, always simplify when possible.

$(\sqrt{3} + 2\sqrt{10})(4\sqrt{3} - \sqrt{10})$

$= 4\sqrt{3} \cdot \sqrt{3} - \sqrt{3} \cdot \sqrt{10} + 8\sqrt{10}\sqrt{3} - 2\sqrt{10}\sqrt{10}$

$= 4 \times 3 - \sqrt{30} + 8\sqrt{30} - 2 \times 10$

$= 12 + 7\sqrt{30} - 20$

$= -8 + 7\sqrt{30}$

$(4\sqrt{13xy})^2$

$= 16 \times 13xy$

$= 208xy$

$(\sqrt{x-1})^2$

$= x - 1$

Problem 2 continued..

$(-2\sqrt[3]{6wz^2})^3$

$= -8 \times 6 w z^2$

$= -48 w z^2$

$(\sqrt{p}-\sqrt{7})^2$

$= (\sqrt{P})^2 - 2\sqrt{P}\sqrt{7} + (\sqrt{7})^2$

$= P + 7 - 2\sqrt{7P}$

$(\sqrt{5}+\sqrt{w})(\sqrt{5}-\sqrt{w})$

$= (\sqrt{5})^2 - \sqrt{5}\sqrt{W} + \sqrt{5}\sqrt{W} - \sqrt{W}\sqrt{W}$

$= 5 - W$

Multiplying Radicals with Different Indices. So far we have been multiplying radicals of the same type, like both being square roots, or both being cube roots, etc. Remember, we can write radicals as fractional exponents. We will use this idea to multiply radicals of different indices.

Example: $\sqrt[3]{5} \cdot \sqrt{5} = 5^{\frac{1}{3}} \cdot 5^{\frac{1}{2}} = 5^{\frac{2}{6}} \cdot 5^{\frac{3}{6}} = 5^{\frac{5}{6}} = \sqrt[6]{5^5}$.

Now you try:

$\sqrt{x} \cdot \sqrt[4]{x}$

$= x^{\frac{1}{2}} \cdot x^{\frac{1}{4}}$

$= x^{\frac{2}{4}} \cdot x^{\frac{1}{4}} = x^{\frac{3}{4}}$

$= \sqrt[4]{x^3}$

$\sqrt[5]{q^4} \cdot \sqrt[3]{q^2}$

$= q^{\frac{4}{5}} \cdot q^{2/3}$

$= q^{\frac{12}{15}} \cdot q^{\frac{10}{15}}$

$= q^{22/15}$

$= \sqrt[15]{q^{22}}$

$\dfrac{\sqrt{u^3}}{\sqrt[3]{u}}$

$= \dfrac{u^{3/2}}{u^{\frac{1}{3}}}$

$= u^{\frac{3}{2}} \cdot u^{-\frac{1}{3}}$

$= u^{\frac{9}{6}} \cdot u^{-\frac{2}{6}}$

$= u^{7/6}$

$= \sqrt[6]{u^7}$

$\dfrac{\sqrt{v^5}}{\sqrt[4]{v}}$

$= \dfrac{v^{\frac{5}{2}}}{v^{\frac{1}{4}}}$

$= v^{\frac{5}{2}} \cdot v^{-\frac{1}{4}}$

$= v^{\frac{10}{4}} \cdot v^{-\frac{1}{4}}$

$= v^{\frac{9}{4}}$

$= \sqrt[4]{v^9}$

Rationalizing the Denominator and Solving Equations with Radicals

1. Simplify the following.

$\sqrt{\frac{4w^3}{25y^4}}$ $= \boxed{\frac{2w\sqrt{w}}{5y^2}}$

$\sqrt[5]{\frac{-32x}{y^{10}}}$ (2^5) $= \boxed{\frac{2\sqrt[5]{-x}}{y^2}}$

$\frac{\sqrt{72ab^5}}{\sqrt{8ab}}$ $= \sqrt{9b^4}$ $= \boxed{3b^2}$

$\frac{\sqrt[3]{128wz^8}}{\sqrt[3]{2wz^2}}$ $= \sqrt[3]{64w^6}$ $= \boxed{4w^2}$

Rationalizing the Denominator is the process of removing a radical from the denominator. To begin this process, we need to identify the missing radicand to be able to take the root.

Example: $\sqrt{a}\cdot\sqrt{?} = \sqrt{a^2} = a$ $\qquad ? = a$

$\sqrt[3]{y}\cdot\sqrt[3]{?} = \sqrt[3]{y^3} = y$ $\qquad ? = y^2$

$\sqrt[5]{2z^3}\cdot\sqrt[5]{?} = \sqrt[3]{2^5z^5} = 2z$ find ?. $\quad \boxed{? = 2^4z^2}$

2. Using your answers from above, rationalize the denominators of the fractions below by multiplying both the numerator and denominator by the correct $\sqrt[n]{?}$ from above.

$\frac{1}{\sqrt{a}} \times \frac{\sqrt{a}}{\sqrt{a}} = \boxed{\frac{\sqrt{a}}{a}}$

$\frac{5}{\sqrt[3]{y}} \times \frac{\sqrt[3]{y^2}}{\sqrt[3]{y^2}} = \boxed{\frac{5\sqrt[3]{y^2}}{y}}$

$\frac{4z^2}{\sqrt[5]{2z^3}} \times \frac{\sqrt[5]{2^4z^2}}{\sqrt[5]{2^4z^2}} = \frac{4z^2 \times \sqrt[5]{2^4z^2}}{2z}$ $= \boxed{2z\sqrt[5]{2^4z^2}}$

Now Try These (Hint: It may help to write the constants in factored form):

$\frac{b^3}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$ $= \frac{b^3\sqrt{3}}{3}$

$\frac{2}{\sqrt{48}} = \frac{2}{\sqrt{16\times 3}}$ $= \frac{2}{4\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \boxed{\frac{\sqrt{3}}{6}}$

$\frac{1}{\sqrt[3]{9}} = \frac{1}{\sqrt[3]{3^2}} \times \frac{\sqrt[3]{3}}{\sqrt[3]{3}}$ $= \boxed{\frac{\sqrt[3]{3}}{3}}$

$\sqrt[3]{\frac{5}{z^2}} = \frac{\sqrt[3]{5}}{\sqrt[3]{z^2}} \times \frac{\sqrt[3]{z}}{\sqrt[3]{z}}$ $= \boxed{\frac{\sqrt[3]{5z}}{z}}$

$\frac{6}{\sqrt[3]{3y^2}} \times \frac{\sqrt[3]{3^2y}}{\sqrt[3]{3^2y}}$ $= \frac{6\sqrt[3]{3^2y}}{3y} = \boxed{\frac{2\sqrt[3]{3^2y}}{y}}$

3. We've seen that $(a+b)(a-b) = a^2 - b^2$, so we will use that to rationalize the denominators with two terms. For example, use this to simplify $(2+\sqrt{6})(2-\sqrt{6})$.

$= 4 - 6 = \boxed{-2}$

Notice your answer no longer has square roots. Use this example to rationalize the denominator of $\frac{-2}{2+\sqrt{6}}$.

$\frac{-2 \times (2-\sqrt{6})}{(2+\sqrt{6})\times(2-\sqrt{6})} = \frac{-4+2\sqrt{6}}{4-6} = \boxed{2-\sqrt{6}}$

Now try these:

$\frac{-12}{\sqrt{5}-3} \frac{\times(\sqrt{5}+3)}{\times(\sqrt{5}+3)}$

$= \frac{-36-12\sqrt{5}}{5-9} = \frac{-36-12\sqrt{5}}{-4}$

$= \boxed{9+3\sqrt{5}}$

$\frac{x-5}{\sqrt{x}+\sqrt{5}} \frac{\times(\sqrt{x}-\sqrt{5})}{\times(\sqrt{x}-\sqrt{5})}$

$= \frac{x\sqrt{x} - x\sqrt{5} - 5\sqrt{x} + 5\sqrt{5}}{x-5}$

$= \frac{\sqrt{x}(x-5) - \sqrt{5}(x-5)}{x-5}$

$= \boxed{\sqrt{x}-\sqrt{5}}$

$\frac{3\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}} \frac{\times(\sqrt{x}-\sqrt{y})}{\times(\sqrt{x}-\sqrt{y})}$

$= \frac{3x - 3\sqrt{xy} - \sqrt{xy} + y}{x-y}$

$= \boxed{\frac{3x-4\sqrt{xy}+y}{x-y}}$

$\frac{3\sqrt{10}}{2+\sqrt{10}} \frac{\times(2-\sqrt{10})}{\times(2-\sqrt{10})}$

$= \frac{6\sqrt{10}-30}{4-10}$

$= \frac{6\sqrt{10}-30}{-6}$

$= \boxed{-\sqrt{10}+5}$

We know that $(\sqrt{x})^2 = x$ for real valued roots, and generally $(\sqrt[n]{x})^n = x$. We can use this to solve radical equations. For example, with $\sqrt[3]{x} = 4$, we can cube both sides to solve for x, $(\sqrt[3]{x})^3 = 4^3$ to get $x = 64$.

However, when we raise both sides to an even power, we might introduce false solutions (since raising to an even power can change the sign). For example, $\sqrt{x} = -7$, when we square both sides to solve for x, we get $x = 49$. Now if we check that solution by plugging $x = 49$ back into the original equation, we get $\sqrt{49} = -7$. This is incorrect, since the principal square root of x must be non-negative. Therefore, **we must check our solutions when raising both sides to an even power.**

4. Solve the following radical equations. please isolate the root first, before raising both sides to a power.

$\sqrt{x} + 4 = 6$

$\sqrt{x} = 6-4$

$(\sqrt{x})^2 = (2)^2$

$\boxed{x = 4}$

check: $\sqrt{4} + 4 = 2+4 = 6$ ✓

$(\sqrt{5y+1})^2 = (4)^2$

$5y + 1 = 16$

$5y = 15$

$y = \frac{15}{5} = \boxed{3}$

check: $\sqrt{15+1} = 4$ ✓

$(2z-3)^{\frac{1}{2}} - 3 = 6$

$(2z-3)^{\frac{1}{2}} = 6+3$

$((2z-3)^{\frac{1}{2}})^2 = (9)^2$

$2z - 3 = 81 \rightarrow 2z = 84$

$\boxed{z = 42}$

check: $(84-3)^{\frac{1}{2}} - 3 = 9 - 3 = 6$ ✓

$(k+18)^{\frac{1}{3}} + 5 = 3$

$(k+18)^{\frac{1}{3}} = 3-5$

$((k+18)^{\frac{1}{3}})^3 = (-2)^3$

$k + 18 = -8 \rightarrow k = -18-8$

$= -26$

$2\sqrt{3-w} - w = 0$

$2\sqrt{3-w} = w$

$(\sqrt{3-w})^2 = (\frac{w}{2})^2$

$3 - w = \frac{w^2}{4} \nearrow \frac{w^2}{4} + w - 3 = 0$

$w = \frac{-1 \pm \sqrt{1 - 4(\frac{1}{4})(-3)}}{2(\frac{1}{4})} = \frac{-1 \pm 2}{\frac{1}{2}}$

-6 ✗ Not in the domain!

$\boxed{2}$ ✓ check: $2\sqrt{3-2} - 2 = 0$ ✓

$\sqrt{5y+1} + 2 = y + 3$

$(\sqrt{5y+1})^2 = (y+1)^2 \rightarrow 5y + 1 = y^2 + 2y + 1$

$\Rightarrow y^2 - 3y = 0 \rightarrow y(y-3) = 0$

$y = 0$ ✓

$y = 3$ ✓

check: $\sqrt{1} + 2 = 0 + 3$ ✓

check: $\sqrt{5+1} + 2 = 3 + 3$

$= 4 + 2 = 6$ ✓

$\sqrt[4]{3b+6} - \sqrt[4]{7b-6} = 0$

$(\sqrt[4]{3b+6})^4 = (\sqrt[4]{7b-6})^4$

$3b + 6 = 7b - 6 \rightarrow 4b = 12$

$\boxed{b = 3}$

check: $\sqrt[4]{9+6} - \sqrt[4]{21-6} = \sqrt[4]{15} - \sqrt[4]{15} = 0$ ✓

$\sqrt{5a-9} = \sqrt{5a} - 3$

$(\sqrt{5a-9})^2 = (\sqrt{5a} - 3)^2$

$\cancel{5a} - 9 = \cancel{5a} + 9 - 2(3)\sqrt{5a}$

$\Rightarrow 6\sqrt{5a} = 18$

$(\sqrt{5a})^2 = (3)^2 \Rightarrow 5a = 9 \Rightarrow \boxed{a = \frac{9}{5}}$

check: $\sqrt{5 \times \frac{9}{5} - 9} = \sqrt{5 \times \frac{9}{5}} - 3 = 0$ ✓

5. The time, $t(d)$, in seconds it takes for an object to drop d meters is given by $t(d) = \sqrt{\frac{d}{4.9}}$. Approximate the height of the Willis Tower in Chicago if it takes an object 9.51 seconds to drop from the top of the tower. Round to the nearest meter.

$t = 9.51\,s \Rightarrow (9.51)^2 = (\sqrt{\frac{d}{4.9}})^2$

$90.44 = \frac{d}{4.9} \Rightarrow d = 443.15 \approx \boxed{443m}$

Complex Numbers

Complex Numbers Definition of i: $i = \sqrt{-1}$, which means that $i^2 = -1$.

$\sqrt{-b} = i\sqrt{b}$ for positive real number b.

1. Simplify the following.

$\sqrt{-81}$	$-\sqrt{-25}$	$\sqrt{-50}$	$-\sqrt{-20}$
$= \sqrt{-1} \cdot \sqrt{81}$	$= -5i$	$= \sqrt{-1} \cdot \sqrt{25 \times 2}$	$= -\sqrt{-1} \cdot \sqrt{4} \cdot \sqrt{5}$
$= 9i$		$= 5\sqrt{2}\, i$	$= -2\sqrt{5}\, i$

2. When simplifying the product or quotient of an imaginary number, first simplify in terms of i, and then perform the multiplication or division.

For Example, $\sqrt{-2} \cdot \sqrt{-18} = i\sqrt{2} \cdot 3i\sqrt{2} = 3i^2 \cdot (\sqrt{2})^2 = 3(-1) \cdot 2 = -6$. Now you try these:

$\sqrt{-9} \cdot \sqrt{-16}$	$\sqrt{-12} \cdot \sqrt{-50}$	$\dfrac{\sqrt{-27}}{\sqrt{9}}$	$\dfrac{\sqrt{-125}}{\sqrt{45}}$
$= 3i \times 4i \quad = 12(-1)$	$i \times i \times \sqrt{3 \times 4} \times \sqrt{2 \times 25}$	$= +i\sqrt{\frac{27}{9}}$	$= \dfrac{i\sqrt{25 \times 5}}{\sqrt{9 \times 5}}$
$= \boxed{-12}$	$= -1 \times 2 \times 5 \times \sqrt{6}$	$= \sqrt{3}\, i$	$= \frac{5}{3} i$
	$= -10\sqrt{6}$		

3. **Powers of i:**

i^n	Decomposed form	Simplified form	i^n	Decomposed form	Simplified form
i	i^1	i	i^5	$i^4 \cdot i = 1 \cdot i$	i
i^2	$i \cdot i$	-1	i^6	$i^4 \cdot i^2 = 1 \cdot -1$	-1
i^3	$i^2 \cdot i = -1 \cdot i$	$-i$	i^7	$i^4 \cdot i^3 = 1 \cdot -i$	$-i$
i^4	$i^2 \cdot i^2 = -1 \cdot -1$	1	i^8	$i^4 \cdot i^4 = 1 \cdot 1$	1

Do you see the pattern? Use that pattern to simplify the following:

i^{13}

$= i^{12} \times i = i^6 \times i^6 \times i$

$= -1 \times -1 \times i = \boxed{i}$

i^{26}

$i^{12} \times i^{12} \times i^2 = 1 \times 1 \times -1 = \boxed{-1}$

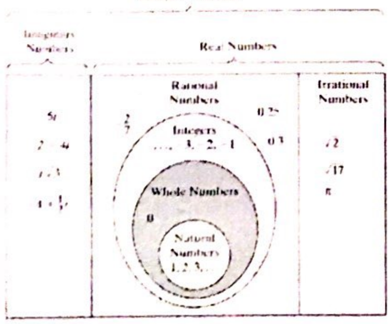
i^{43}

$= i^{20} \times i^{20} \times i^2 \times i$

$= 1 \times 1 \times -1 \times i = \boxed{-i}$

Definition of Complex Numbers: A complex number is a number of the form $a + bi$, where a and b are real numbers, and $i = \sqrt{-1}$.

If $b = 0$, the complex number is also a real number. If $b \neq 0$, it is an imaginary number. $a + bi$ and $a - bi$ are called complex conjugates.



4. Addition/Subtraction of complex numbers. $(a + bi) \pm (c + di) = (a \pm c) + (b \pm d)i$, in other words, like we have been doing for like terms, add/subtract the real parts and add/subtract the imaginary parts, keeping them separate. Try the following:

$(2 - i) + (5 + 7i)$

$= 7 - i + 7i$

$= \boxed{7 + 6i}$

$-\frac{7}{5}i - (-\frac{2}{5} + \frac{3}{5}i)$

$= (-\frac{7}{5} - \frac{3}{5})i + \frac{2}{5}$

$= (\frac{-10}{5})i + \frac{2}{5}$

$= \boxed{-2i + \frac{2}{5}}$

$(2 + 3i) - (1 - 4i) + (-2 + 7i)$

$2 - 1 - 2 + 3i + 4i + 7i$

$= \boxed{-1 + 14i}$

5. Multiplication of complex numbers.

$6i(1 - 3i)$

$6i - 18(i)^2$

$= \boxed{6i + 18}$

$(2 - 10i)(3 + 2i)$

$= 6 + 4i - 30i - 20i^2$

$= 6 - 26i + 20$

$= \boxed{26 - 26i}$

$(4 + 5i)^2$

$16 + 25i^2 + 2(4)(5i)$

$= 16 - 25 + 40i$

$= \boxed{-9 + 40i}$

$(5 + 2i)(5 - 2i)$

$25 - 10i + 10i - 4i^2$

$= 25 + 4 = \boxed{29}$

6. For division of complex numbers, we multiply the numerator and denominator of the fraction by the complex conjugate of the denominator. Try the following. Write your answer in $a + bi$ form.

$\frac{2}{1+3i} \frac{(1-3i)}{(1-3i)}$

$\frac{2-6i}{1-9i^2} = \frac{2-6i}{1+9} = \frac{2-6i}{10}$

$= \boxed{\frac{1}{5} - \frac{3}{5}i}$

$\frac{-i}{4-3i} \frac{(4+3i)}{(4+3i)}$

$= \frac{-4i - 3i^2}{16 - 9i^2}$

$= \frac{-4i+3}{16+9} = \frac{-4i+3}{25}$

$= \boxed{-\frac{4}{25}i + \frac{3}{25}}$

$\frac{7+3i}{4-2i} \times \frac{(4+2i)}{(4+2i)}$

$= \frac{28 + 14i + 12i + 6i^2}{16 - 4i^2}$

$= \frac{28 - 6 + 26i}{16 + 4}$

$= \frac{22 + 26i}{20} = \frac{11 + 13i}{10}$

$= \boxed{\frac{11}{10} + \frac{13}{10}i}$

$\frac{-6-i}{-i} \times \frac{i}{i}$

$= \frac{-6i - i^2}{-i^2}$

$= \boxed{-6i + 1}$

7. Simplify the following and write in $a + bi$ form.

$\frac{2 + \sqrt{-16}}{8}$

$\frac{2 + i4}{8} = \boxed{\frac{1}{4} + \frac{1}{2}i}$

$\frac{-6 + \sqrt{-72}}{6}$

$= -1 + \frac{i\sqrt{9 \times 8}}{6}$

$= -1 + \frac{i\sqrt{3^2 \times 2^3}}{6}$

$= -1 + \frac{i\,6\sqrt{2}}{6}$

$= \boxed{-1 + \sqrt{2}i}$

$\frac{-5 + \sqrt{-50}}{10}$

$= -\frac{1}{2} + \frac{i\sqrt{25 \times 2}}{10}$

$= -\frac{1}{2} + \frac{5\sqrt{2}i}{10}$

$= \boxed{\frac{-1}{2} + \frac{\sqrt{2}}{2}i}$